

# Strictly Implicit Priority Queues: On the Number of Moves and Worst-Case Time

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**Abstract.** The binary heap of Williams (1964) is a simple priority queue characterized by only storing an array containing the elements and the number of elements  $n$  – here denoted a *strictly implicit* priority queue. We introduce two new strictly implicit priority queues. The first structure supports amortized  $O(1)$  time INSERT and  $O(\log n)$  time EXTRACTMIN operations, where both operations require amortized  $O(1)$  element moves. No previous implicit heap with  $O(1)$  time INSERT supports both operations with  $O(1)$  moves. The second structure supports worst-case  $O(1)$  time INSERT and  $O(\log n)$  time (and moves) EXTRACTMIN operations. Previous results were either amortized or needed  $O(\log n)$  bits of additional state information between operations.

## 1 Introduction

In 1964 Williams presented “Algorithm 232” [12], commonly known as the binary heap. The binary heap is a priority queue data structure storing a dynamic set of  $n$  elements from a totally ordered universe, supporting the insertion of an element (INSERT) and the deletion of the minimum element (EXTRACTMIN) in worst-case  $O(\log n)$  time. The binary heap structure is an *implicit* data structure, i.e., it consists of an array of length  $n$  storing the elements, and no information is stored between operations except for the array and the value  $n$ . Sometimes data structures storing  $O(1)$  additional words are also called implicit. In this paper we restrict our attention to *strictly implicit* priority queues, i.e., data structures that do not store any additional information than the array of elements and the value  $n$  between operations.

Due to the  $\Omega(n \log n)$  lower bound on comparison based sorting, either INSERT or EXTRACTMIN must take  $\Omega(\log n)$  time, but not necessarily both. Carlson *et al.* [5] presented an implicit priority queue with worst-case  $O(1)$  and  $O(\log n)$  time INSERT and EXTRACTMIN operations, respectively. However, the structure is not strictly implicit since it needs to store  $O(1)$  additional words. Harvey and Zatloukal [11] presented a strictly implicit priority structure achieving the same bounds, but amortized. No previous strictly implicit priority queue with matching worst-case time bounds is known.

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\* Work supported in part by the Danish National Research Foundation grant DNRF84 through the Center for Massive Data Algorithmics.

**Table 1.** Selected previous and new results for implicit priority queues. The bounds are asymptotic, and  $\star$  are amortized bounds.

	EXTRACT-			Strict	Identical elements
	INSERT	MIN	Moves		
Williams [12]	$\log n$	$\log n$	$\log n$	yes	yes
Carlsson <i>et al.</i> [5]	1	$\log n$	$\log n$	no	yes
Edelkamp <i>et al.</i> [7]	1	$\log n$	$\log n$	no	yes
Harvey and Zatloukal [11]	$\star 1$	$\star \log n$	$\star \log n$	yes	yes
Franceschini and Munro [9]	$\star \log n$	$\star \log n$	$\star 1$	yes	no
Section 2	$\star 1$	$\star \log n$	$\star 1$	yes	yes
Section 3	1	$\log n$	$\log n$	yes	no

A measurement often studied in implicit data structures and in-place algorithms is the number of element moves performed during the execution of a procedure. Franceschini showed how to sort  $n$  elements implicitly using  $O(n \log n)$  comparisons and  $O(n)$  moves [8], and Franceschini and Munro [9] presented implicit dictionaries with amortized  $O(\log n)$  time updates with amortized  $O(1)$  moves per update. The latter immediately implies an implicit priority queue with amortized  $O(\log n)$  time INSERT and EXTRACTMIN operations performing amortized  $O(1)$  moves per operation. No previous implicit priority queue with  $O(1)$  time INSERT operations achieving  $O(1)$  moves per operation is known.

For a more thorough survey of previous priority queue results, see [1].

*Our Contribution* We present two strictly implicit priority queues. The first structure (Section 2) limits the number of moves to  $O(1)$  per operation with amortized  $O(1)$  and  $O(\log n)$  time INSERT and EXTRACTMIN operations, respectively. However the bounds are all amortized and it remains an open problem to achieve these bounds in the worst case for strictly implicit priority queues. We note that this structure implies a different way of sorting in-place with  $O(n \log n)$  comparisons and  $O(n)$  moves. The second structure (Section 3) improves over [5, 11] by achieving INSERT and EXTRACTMIN operations with worst-case  $O(1)$  and  $O(\log n)$  time (and moves), respectively. The structure in Section 3 assumes all elements to be distinct where as the structure in Section 2 also can be extended to support identical elements (see [4]). See Figure 1 for a comparison of new and previous results.

*Preliminaries* We assume the *strictly implicit model* as defined in [3] where we are only allowed to store the number of elements  $n$  and an array containing the  $n$  elements. Comparisons are the only allowed operations on the elements. The number  $n$  is stored in a memory cell with  $\Theta(\log n)$  bits (word size) and any operation usually found in a RAM is allowed for computations on  $n$  and intermediate values. The number of moves is the number of writes to the array storing the elements. That is, swapping two elements costs two moves.

A fundamental technique in the implicit model is to encode a 0/1-bit with a pair of distinct elements  $(x, y)$ , where the pair encodes 1 if  $x < y$  and 0 otherwise.

A binary heap is a complete binary tree structure where each node stores an element and the tree satisfies *heap order*, i.e., the element at a non-root node is larger than or equal to the element at the parent node. Binary heaps can be generalized to  $d$ -ary heaps [10], where the degree of each node is  $d$  rather than two. This implies  $O(\log_d n)$  and  $O(d \log_d n)$  time for INSERT and EXTRACTMIN, respectively, using  $O(\log_d n)$  moves for both operations.

## 2 Amortized $O(1)$ moves

In this section we describe a strictly implicit priority queue supporting amortized  $O(1)$  time INSERT and amortized  $O(\log n)$  time EXTRACTMIN. Both operations perform amortized  $O(1)$  moves. In Sections 2.1-2.3 we assume elements are distinct. In [4] we describe how to handle identical elements.

*Overview* The basic idea of our priority queue is the following (the details are presented in Section 2.1). The structure consists of four components: an insertion buffer  $B$  of size  $O(\log^3 n)$ ;  $m$  insertion heaps  $I_1, I_2, \dots, I_m$  each of size  $\Theta(\log^3 n)$ , where  $m = O(n/\log^3 n)$ ; a singles structure  $T$ , of size  $O(n)$ ; and a binary heap  $Q$ , storing  $\{1, 2, \dots, m\}$  (integers encoded by pairs of elements) with the ordering  $i \leq j$  if and only if  $\min I_i \leq \min I_j$ . Each  $I_i$  and  $B$  is a  $\log n$ -ary heap of size  $O(\log^3 n)$ . The table below summarizes the performance of each component:

Structure	Insert		ExtractMin	
	Time	Moves	Time	Moves
$B, I_i$	1	1	$\log n$	1
$Q$	$\log^2 n$	$\log^2 n$	$\log^2 n$	$\log^2 n$
$T$	$\log n$	1	$\log n$	1

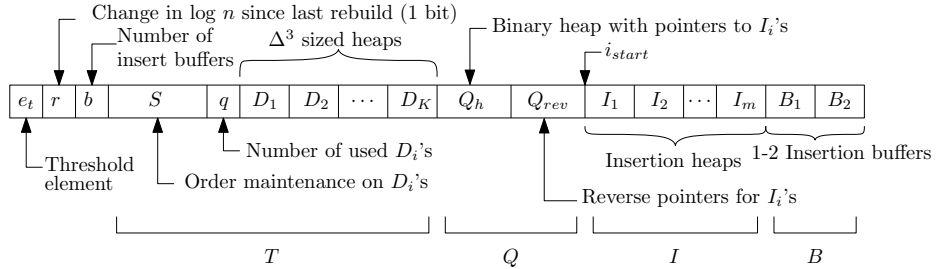
It should be noted that the implicit dictionary of Franceschini and Munro [9] could be used for  $T$ , but we will give a more direct solution since we only need the restricted EXTRACTMIN operation for deletions.

The INSERT operation inserts new elements into  $B$ . If the size of  $B$  becomes  $\Theta(\log^3 n)$ , then  $m$  is incremented by one,  $B$  becomes  $I_m$ ,  $m$  is inserted into  $Q$ , and  $B$  becomes a new empty  $\log n$ -ary heap. An EXTRACTMIN operation first identifies the minimum element in  $B$ ,  $Q$  and  $T$ . If the overall minimum element  $e$  is in  $B$  or  $T$ ,  $e$  is removed from  $B$  or  $T$ . If the minimum element  $e$  resided in  $I_i$ , where  $i$  is stored at the root of  $Q$ , then  $e$  and  $\log^2 n$  further smallest elements are extracted from  $I_i$  (if  $I_i$  is not empty) and all except  $e$  inserted into  $T$  ( $T$  has cheap operations whereas  $Q$  does not, thus the expensive operation on  $Q$  is amortized over inexpensive ones in  $T$ ), and  $i$  is deleted from and reinserted into  $Q$  with respect to the new minimum element in  $I_i$ . Finally  $e$  is returned.

For the analysis we see that INSERT takes  $O(1)$  time and moves, except when converting  $B$  to a new  $I_m$  and inserting  $m$  into  $Q$ . The  $O(\log^2 n)$  time and moves for this conversion is amortized over the insertions into  $B$ , which becomes amortized  $O(1)$ , since  $|B| = \Omega(\log^2 n)$ . For EXTRACTMIN we observe that an expensive deletion from  $Q$  only happens once for every  $\log^2 n$ -th element

from  $I_i$  (the remaining ones from  $I_i$  are moved to  $T$  and deleted from  $T$ ), and finally if there have been  $d$  EXTRACTMIN operations, then at most  $d + m \log^2 n$  elements have been inserted into  $T$ , with a total cost of  $O((d + m \log^2 n) \log n) = O(n + d \log n)$ , since  $m = O(n / \log^3 n)$ .

## 2.1 The implicit structure



**Fig. 1.** The different structures and their layout in memory.

We now give the details of our representation (see Figure 1). We select one element  $e_t$  as our *threshold element*, and denote elements greater than  $e_t$  as *dummy elements*. The current number of elements in the priority queue is denoted  $n$ . We fix an integer  $N$  that is an approximation of  $n$ , where  $N \leq n < 4N$  and  $N = 2^j$  for some  $j$ . Instead of storing  $N$ , we store a bit  $r = \lfloor \log n \rfloor - \log N$ , encoded by two dummy elements. We can then compute  $N$  as  $N = 2^{\lfloor \log n \rfloor - r}$ , where  $\lfloor \log n \rfloor$  is the position of the most significant bit in the binary representation of  $n$  (which we assume is computable in constant time). The value  $r$  is easily maintained: When  $\lfloor \log n \rfloor$  changes,  $r$  changes accordingly. We let  $\Delta = \log(4N) = \lfloor \log n \rfloor + 2 - r$ , i.e.,  $\Delta$  bits is sufficient to store an integer in the range  $0..n$ . We let  $M = \lceil 4N/\Delta^3 \rceil$ .

We maintain the invariant that the size of the insertion buffer  $B$  satisfies  $1 \leq |B| \leq 2\Delta^3$ , and that  $B$  is split into two parts  $B_1$  and  $B_2$ , each being  $\Delta$ -ary heaps ( $B_2$  possibly empty), where  $|B_1| = \min\{|B|, \Delta^3\}$  and  $|B_2| = |B| - |B_1|$ . We use two buffers to prevent expensive operation sequences that alternate inserting and deleting the same element. We store a bit  $b$  indicating if  $B_2$  is nonempty, i.e.,  $b = 1$  if and only if  $|B_2| \neq 0$ . The bit  $b$  is encoded using two dummy elements. The structures  $I_1, I_2, \dots, I_m$  are  $\Delta$ -ary heaps storing  $\Delta^3$  elements. The binary heap  $Q$  is stored using two arrays  $Q_h$  and  $Q_{rev}$  each of a fixed size  $M \geq m$  and storing integers in the range  $1..m$ . Each value in both arrays is encoded using  $2\Delta$  dummy elements, i.e.,  $Q$  is stored using  $4M\Delta$  dummy elements. The first  $m$  entries of  $Q_h$  store the binary heap, whereas  $Q_{rev}$  acts as reverse pointers, i.e., if  $Q_h[j] = i$  then  $Q_{rev}[i] = j$ . All operations on a regular binary heap take  $O(\log n)$  time, but since each “read”/“write” from/to  $Q$  needs to decode/encode

an integer the time increases by a factor  $2\Delta$ . It follows that  $Q$  supports INSERT and EXTRACTMIN in  $O(\log^2 n)$  time, and FINDMIN in  $O(\log n)$  time.

We now describe  $T$  and we need the following density maintenance result.

**Lemma 1 ([2]).** *There is a dynamic data structure storing  $n$  comparable elements in an array of length  $(1 + \varepsilon)n$ , supporting INSERT and EXTRACTMIN in amortized  $O(\log^2 n)$  time and FINDPREDECESSOR in worst case  $O(\log n)$  time. FINDPREDECESSOR does not modify the array.*

**Corollary 1.** *There is an implicit data structure storing  $n$  (key, index) pairs, while supporting INSERT and EXTRACTMIN in amortized  $O(\log^3 n)$  time and moves, and FINDPREDECESSOR in  $O(\log n)$  time in an array of length  $\Delta(2 + \varepsilon)n$ .*

*Proof.* We use the structure from Lemma 1 to store pairs of a key and an index, where the index is encoded using  $2\Delta$  dummy elements. All additional space is filled with dummy elements. However comparisons are only made on keys and not indexes, which means we retain  $O(\log n)$  time for FINDMIN. Since the stored elements are now an  $O(\Delta) = \Theta(\log n)$  factor larger, the time for update operations becomes an  $O(\log n)$  factor slower giving amortized  $O(\log^3 n)$  time for INSERT and EXTRACTMIN.  $\square$

The singles structure  $T$  intuitively consists of a sorted list of the elements stored in  $T$  partitioned into buckets  $D_1, \dots, D_q$  of size at most  $\Delta^3$ , where the minimum element  $e$  from bucket  $D_i$  is stored in a structure  $S$  from Corollary 1 as the pair  $(e, i)$ . Each  $D_i$  is stored as a  $\Delta$ -ary heap of size  $\Delta^3$ , where empty slots are filled with dummy elements. Recall implicit heaps are complete trees, which means all dummy elements in  $D_i$  are stored consecutively after the last non-dummy element. In  $S$  we consider pairs  $(e, i)$  where  $e > e_t$  to be empty spaces.

More specifically, the structure  $T$  consists of:  $q, S, D_1, D_2, \dots, D_K$ , where  $K = \lceil \frac{N}{16\Delta^3} \rceil \geq q$  is the number of  $D_i$ 's available. The structure  $S$  uses  $\lceil \frac{N}{4\Delta^2} \rceil$  elements and  $q$  uses  $2\Delta$  elements to encode a pointer. Each  $D_i$  uses  $\Delta^3$  elements.

The  $D_i$ 's and  $S$  relate as follows. The number of  $D_i$ 's is at most the maximum number of items that can be stored in  $S$ . Let  $(e, i) \in S$ , then  $\forall x \in D_i : e < x$ , and furthermore for any  $(e', i') \in S$  with  $e < e'$  we have  $\forall x \in D_i : x < e'$ . These invariants do not apply to dummy elements. Since  $D_i$  is a  $\Delta$ -ary heap with  $\Delta^3$  elements we get  $O(\log_\Delta \Delta^3) = O(1)$  time for INSERT and  $O(\Delta \log_\Delta \Delta^3) = O(\Delta)$  for EXTRACTMIN on a  $D_i$ .

## 2.2 Operations

For both INSERT and EXTRACTMIN we need to know  $N, \Delta$ , and whether there are one or two insert buffers as well as their sizes. First  $r$  is decoded and we compute  $\Delta = 2 + \text{msb}(n) - r$ , where  $\text{msb}(n)$  is the position of the most significant bit in the binary representation of  $n$  (indexed from zero). From this we compute  $N = 2^{\Delta-2}$ ,  $K = \lceil N/(16\Delta^3) \rceil$ , and  $M = \lceil 4N/\Delta^3 \rceil$ . By decoding  $b$  we get the number of insert buffers. To find the sizes of  $B_1$  and  $B_2$  we compute the value  $i_{start}$  which is the index of the first element in  $I_1$ . The size of  $B_1$  is computed as follows. If  $(n - i_{start}) \bmod \Delta^3 = 0$  then  $|B_1| = \Delta^3$ . If  $B_2$  exists then  $B_1$

starts at  $n - 2\Delta^3$  and otherwise  $B_1$  starts at  $n - \Delta^3$ . If  $B_2$  exists and  $(n - i_{start}) \bmod \Delta^3 = 0$  then  $|B_2| = \Delta^3$ , otherwise  $|B_2| = (n - i_{start}) \bmod \Delta^3$ . Once all of this information is computed the actual operation can start. If  $n = N + 1$  and an EXTRACTMIN operation is called, then the EXTRACTMIN procedure is executed and afterwards the structure is rebuilt as described in the paragraph below. Similarly if  $n = 4N - 1$  before an INSERT operation the new element is appended and the data structure is rebuilt.

INSERT If  $|B_1| < \Delta^3$  the new element is inserted in  $B_1$  by the standard insertion algorithm for  $\Delta$ -ary heaps. If  $|B_1| = \Delta^3$  and  $|B_2| = 0$  and a new element is inserted the two elements in  $b$  are swapped to indicate that  $B_2$  now exists. When  $|B_1| = |B_2| = \Delta^3$  and a new element is inserted,  $B_1$  becomes  $I_{m+1}$ ,  $B_2$  becomes  $B_1$ ,  $m + 1$  is inserted in  $Q$  (possibly requiring  $O(\log n)$  values in  $Q_h$  and  $Q_{rev}$  to be updated in  $O(\log^2 n)$  time). Finally the new element becomes  $B_2$ .

EXTRACTMIN Searches for the minimum element  $e$  are performed in  $B_1$ ,  $B_2$ ,  $S$ , and  $Q$ . If  $e$  is in  $B_1$  or  $B_2$  it is deleted, the last element in the array is swapped with the now empty slot and the usual bubbling for heaps is performed. If  $B_2$  disappears as a result, the bit  $b$  is updated accordingly. If  $B_1$  disappears as a result,  $I_m$  becomes  $B_1$ , and  $m$  is removed from  $Q$ .

If  $e$  is in  $I_i$  then  $i$  is deleted from  $Q$ ,  $e$  is extracted from  $I_i$ , and the last element in the array is inserted in  $I_i$ . The  $\Delta^2$  smallest elements in  $I_i$  are extracted and inserted into the *singles structure*: for each element a search in  $S$  is performed to find the range it belongs to, i.e.  $D_j$ , the structure it is to be inserted in. Then it is inserted in  $D_j$  (replacing a dummy element that is put in  $I_i$ , found by binary search). If  $|D_j| = \Delta^3$  and  $q = K$  the priority queue is rebuilt. Otherwise if  $|D_j| = \Delta^3$ ,  $D_j$  is split in two by finding the median  $y$  of  $D_j$  using a linear time selection algorithm [6]. Elements  $\geq y$  in  $D_j$  are swapped with the first  $\Delta^3/2$  elements in  $D_q$  then  $D_j$  and  $D_q$  are made into  $\Delta$ -ary heaps by repeated insertion. Then  $y$  is extracted from  $D_q$  and  $(y, q)$  is inserted in  $S$ . The dummy element pushed out of  $S$  by  $y$  is inserted in  $D_q$ . Finally  $q$  is incremented and we reinsert  $i$  into  $Q$ . Note that it does not matter if any of the elements in  $I_i$  are dummy elements, the invariants are still maintained.

If  $(e, i) \in S$ , the last element of the array is inserted into the singles structure, which pushes out a dummy element  $z$ . The minimum element  $y$  of  $D_i$  is extracted and  $z$  inserted instead. We replace  $e$  by  $y$  in  $S$ . If  $y$  is a dummy element, we update  $S$  as if  $(y, i)$  was removed. Finally  $e$  is returned. Note this might make  $B_1$  or  $B_2$  disappear as a result and the steps above are executed if needed.

*Rebuilding* We let the new  $N = n'/2$ , where  $n'$  is  $n$  rounded to the nearest power of two. Using a linear time selection algorithm [6], find the element with rank  $n - i_{start}$ , this element is the new threshold element  $e_t$ , and it is put in the first position of the array. Following  $e_t$  are all the elements greater than  $e_t$  and they are followed by all the elements comparing less than  $e_t$ . We make sure to have at least  $\Delta^3/2$  elements in  $B_1$  and at most  $\Delta^3/2$  elements in  $B_2$  which dictates whether  $b$  encodes 0 or 1. The value  $q$  is initialized to 1. All the  $D_i$  structures are considered empty since they only contain dummy elements. The pointers in

$Q_h$  and  $Q_{rev}$  are all reset to the value 0. All the  $I_i$  structures as well as  $B_1$  (and possibly  $B_2$ ) are made into  $\Delta$ -ary heaps with the usual heap construction algorithm. For each  $I_j$  structure the  $\Delta^2$  smallest elements are inserted in the singles structure as described in the EXTRACTMIN procedure, and  $j$  is inserted into  $Q$ . The structure now satisfies all the invariants.

### 2.3 Analysis

In this subsection we give the analysis that leads to the following theorem.

**Theorem 1.** *There is a strictly implicit priority queue supporting INSERT in amortized  $O(1)$  time, EXTRACTMIN in amortized  $O(\log n)$  time. Both operations perform amortized  $O(1)$  moves.*

**INSERT** While  $|B| < 2\Delta^3$ , each insertion takes  $O(1)$  time. When an insertion happens and  $|B| = 2\Delta^3$ , the insertion into  $Q$  requires  $O(\log^2 n)$  time and moves. During a sequence of  $s$  insertions, this can at most happen  $\lceil s/\Delta^3 \rceil$  times, since  $|B|$  can only increase for values above  $\Delta^3$  by insertions, and each insertion at most causes  $|B|$  to increase by one. The total cost for  $s$  insertions is  $O(s + s/\Delta^3 \cdot \log^2 n) = O(s)$ , i.e., amortized constant per insertion.

**EXTRACTMIN** We first analyze the cost of updating the singles structure. Each operation on a  $D_i$  takes time  $O(\Delta)$  and performs  $O(1)$  moves. Locating an appropriate bucket using  $S$  takes  $O(\log n)$  time and no moves. At least  $\Omega(\Delta^3)$  operations must be performed on a bucket to trigger an expensive bucket split or bucket elimination in  $S$ . Since updating  $S$  takes  $O(\log^3 n)$  time, the amortized cost for updating  $S$  is  $O(1)$  moves per insertion and extraction from the singles structure. In total the operations on the singles structure require amortized  $O(\log n)$  times and amortized  $O(1)$  moves. For EXTRACTMIN the searches performed all take  $O(\log n)$  comparisons and no moves. If  $B_1$  disappears as a result of an extraction we know at least  $\Omega(\Delta^3)$  extractions have occurred because a rebuild ensures  $|B_1| \geq \Delta^3/2$ . These extractions pay for extracting  $I_m$  from  $Q_h$  which takes  $O(\log^2 n)$  time and moves, amortized this gives  $O(1/\log n)$  additional time and moves. If the extracted element was in  $I_i$  for some  $i$ , then  $\Delta^2$  insertions occur in the singles structure each taking  $O(\log n)$  time and  $O(1)$  moves amortized. If that happens either  $\Omega(\Delta^3)$  insertions or  $\Delta^2$  extractions have occurred: Suppose no elements from  $I_i$  have been inserted in the singles structure, then the reason there is a pointer to  $I_i$  in  $Q_h$  is due to  $\Omega(\Delta^3)$  insertions. When inserting elements in the singles structure from  $I_i$  the number of elements inserted is  $\Delta^2$  and these must first be deleted. From this discussion it is evident that we have saved up  $\Omega(\Delta^2)$  moves and  $\Omega(\Delta^3)$  time, which pay for the expensive extraction. Finally if the minimum element was in  $S$ , then an extraction on a  $\Delta$ -ary heap is performed which takes  $O(\Delta)$  time and  $O(1)$  moves, since its height is  $O(1)$ .

*Rebuilding* The cost of rebuilding is  $O(n)$ , due to a selection and building heaps with  $O(1)$  height. There are three reasons a rebuild might occur: (i)  $n$  became

$4N$ , (ii)  $n$  became  $N - 1$ , or (iii) An insertion into  $T$  would cause  $q > K$ . By the choice of  $N$  during a rebuild it is guaranteed that in the first and second case at least  $\Omega(N)$  insertions or extractions occurred since the last rebuild, and we have thus saved up at least  $\Omega(N)$  time and moves. For the last case we know that each extraction incur  $O(1)$  insertions in the singles structure in an amortized sense. Since the singles structure accommodates  $\Omega(N)$  elements and a rebuild ensures the singles structure has  $o(n)$  non dummy elements (Lemma 2), at least  $\Omega(N)$  extractions have occurred which pay for the rebuild.

**Lemma 2.** *Immediately after a rebuild  $o(n)$  elements in the singles structure are non-dummy elements*

*Proof.* There are at most  $n/\Delta^3$  of the  $I_i$  structures and  $\Delta^2$  elements are inserted in the singles structure from each  $I_i$ , thus at most  $n/\Delta = o(n)$  non-dummy elements reside in the singles structure after a rebuild.  $\square$

The paragraphs above establish Theorem 1.

### 3 Worst case solution

In this section we present a strictly implicit priority queue supporting INSERT in worst-case  $O(1)$  time and EXTRACTMIN in worst-case  $O(\log n)$  time (and moves). The data structure requires all elements to be distinct. The main concept used is a variation on binomial trees. The priority queue is a forest of  $O(\log n)$  such trees. We start with a discussion of the variant we call *relaxed binomial trees*, then we describe how to maintain a forest of these trees in an amortized sense, and finally we give the deamortization.

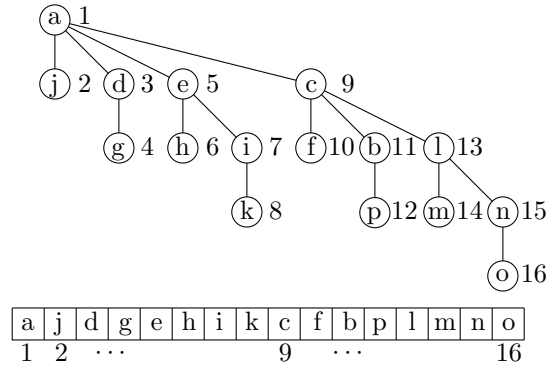
#### 3.1 Relaxed binomial tree

*Binomial trees* are defined inductively: A single node is a binomial tree of size one and the node is also the root. A binomial tree of size  $2^{i+1}$  is made by *linking* two binomial trees  $T_1$  and  $T_2$  both of size  $2^i$ , such that one root becomes the rightmost child of the other root. We lay out in memory a binomial tree of size  $2^i$  by a preorder traversal of the tree where children are visited in order of increasing size, i.e.  $c_0, c_1, \dots, c_{i-1}$ . This layout is also described in [5]. See Figure 2 for an illustration of the layout. In a *relaxed binomial tree* (RBT) each nodes stores an element, satisfying the following order: Let  $p$  be a node with  $i$  children, and let  $c_j$  be a child of  $p$ . Let  $T_{c_j}$  denote the set of elements in the subtree rooted at  $c_j$ . We have the invariant that the element  $c_\ell$  is less than either all elements in  $T_{c_\ell}$  or less than all elements in  $\bigcup_{j < \ell} T_{c_j}$  (see Figure 2). In particular we have the requirement that the root must store the smallest element in the tree. In each node we store a flag indicating in which direction the ordering is satisfied. Note that linking two adjacent RBTs of equal size can be done in  $O(1)$  time: compare the keys of the two roots, if the lesser is to the right, swap the two nodes and finally update the flags to reflect the changes as just described.

For an unrelated technical purpose we also need to store whether a node is the root of a RBT. This information is encoded using three elements per node



(allowing  $3! = 6$  permutations, and we only need to differentiate between three states per node: “root”, “minimum of its own subtree”, or “minimum among strictly smaller subtrees”).



**Fig. 2.** An example of an RBT on 16 elements (a,b,...,o). The layout in memory of an RBT and a regular binomial tree is the same. Note here that node 9 has element  $c$  and is not the minimum of its subtree because node 11 has element  $b$ , but  $c$  is the minimum among the subtrees rooted at nodes 2, 3, and 5 ( $c_0$ ,  $c_1$ , and  $c_2$ ). Note also that node 5 is the minimum of its subtree but not the minimum among the trees rooted at nodes 2 and 3, which means only one state is valid. Finally node 3 is the minimum of both its own subtree and the subtree rooted at node 2, which means both states are valid for that node.

To extract the minimum element of an RBT it is replaced by another element. The reason for replacing is that the forest of RBTs is implicitly maintained in an array and elements are removed from the right end, meaning only an element from the last RBT is removed. If the last RBT is of size 1, it is trivial to remove the element. If it is larger, then we *decompose* it. We first describe how to perform a DECOMPOSE operation which changes an RBT of size  $2^i$  into  $i$  structures  $T_{i-1}, \dots, T_1, T_0$ , where  $|T_j| = 2^j$ . Then we describe how to perform REPLACEMIN which takes one argument, a new element, and extracts the minimum element from an RBT and inserts the argument in the same structure.

A DECOMPOSE procedure is essentially reversing insertions. We describe a tail recursive procedure taking as argument a node  $r$ . If the structure is of size one, we are done. If the structure is of size  $2^i$  the  $(i - 1)$ th child,  $c_{i-1}$ , of  $r$  is inspected, if it is not the minimum of its own subtree, the element of  $c_{i-1}$  and  $r$  are swapped. The  $(i - 1)$ th child should now encode “root”, that way we have two trees of size  $2^{i-1}$  and we recurse on the subtree to the right in the memory layout. This procedure terminates in  $O(i)$  steps and gives  $i + 1$  structures of sizes  $2^{i-1}, 2^{i-2}, \dots, 2, 1$ , and 1 laid out in decreasing order of size (note there are two structures of size 1). This enables easy removal of a single element.

The REPLACEMIN operation works similarly to the DECOMPOSE, where instead of always recursing on the right, we recurse where the minimum element is

the root. When the recursion ends, the minimum element is now in a structure of size 1, which is deleted and replaced by the new element. The decomposition is then reversed by linking the RBTs using the LINK procedure. Note it is possible to keep track of which side was recursed on at every level with  $O(\log n)$  extra bits, i.e.  $O(1)$  words. The operation takes  $O(\log n)$  steps and correctness follows by the DECOMPOSE and LINK procedures. This concludes the description of RBTs and yields the following theorem.

**Theorem 2.** *On an RBT with  $3 \cdot 2^i$  elements, LINK and FINDMIN can be supported in  $O(1)$  time and DECOMPOSE and REPLACEMIN in  $O(i)$  time.*

### 3.2 How to maintain a forest

As mentioned our priority queue is a forest of the relaxed binomial trees from Theorem 2. An easy amortized solution is to store one structure of size  $3 \cdot 2^j$  for every set bit  $j$  in the binary representation of  $\lfloor n/3 \rfloor$ . During an insertion this could cause  $O(\log n)$  LINK operations, but by a similar argument to that of binary counting, this yields  $O(1)$  amortized insertion time. We are aiming for a worst case constant time solution so we maintain the invariant that there are at most 5 structures of size  $2^i$  for  $i = 0, 1, \dots, \lfloor \log n \rfloor$ . This enables us to postpone some of the LINK operations to appropriate times. We are storing  $O(\log n)$  RBTs, but we do not store which sizes we have, this information must be decodable in constant time since we do not allow storing additional words. Recall that we need 3 elements per node in an RBT, thus in the following we let  $n$  be the number of elements and  $N = \lfloor n/3 \rfloor$  be the number of nodes. We say a node is in node position  $k$  if the three elements in it are in positions  $3k - 2$ ,  $3k - 1$ , and  $3k$ . This means there is a buffer of 0, 1, or 2 elements at the end of the array. When a third element is inserted, the elements in the buffer become an RBT with a single node and the buffer is now empty. If an INSERT operation does not create a new node, the new element is simply appended to the buffer. We are not storing the structure of the forest (i.e. how many RBTs of size  $2^j$  exists for each  $j$ ), since that would require additional space. To be able to navigate the forest we need the following two lemmas.

**Lemma 3.** *There is a structure of size  $2^i$  at node positions  $k, k+1, \dots, k+2^i-1$  if and only if the node at position  $k$  encodes “root”, the node at position  $k+2^i$  encodes “root” and the node at position  $k+2^{i-1}$  encodes “not root”.*

*Proof.* It is trivially true that the mentioned nodes encode “root”, “root” and “not root” if an RBT with  $2^i$  nodes is present in those locations.

We first observe there cannot be a structure of size  $2^{i-1}$  starting at position  $k$ , since that would force the node at position  $k+2^{i-1}$  to encode “root”. Also all structures between  $k$  and  $N$  must have less than  $2^i$  elements, since both nodes at positions  $k$  and  $k+2^i$  encode “root”. We now break the analysis in a few cases and the lemma follows from a proof by contradiction. Suppose there is a structure of size  $2^{i-2}$  starting at  $k$ , then for the same reason as before there cannot be another one of size  $2^{i-2}$ . Similarly, there can at most be one structure of size  $2^{i-3}$  following that structure. Now we can bound the total number of

nodes from position  $k$  onwards in the structure as:  $2^{i-2} + 2^{i-3} + 5 \sum_{j=0}^{i-4} 2^j = 2^i - 5 < 2^i$ , which is a contradiction. So there cannot be a structure of size  $2^{i-2}$  starting at position  $k$ . Note there can at most be three structures of size  $2^{i-3}$  starting at position  $k$ , and we can again bound the total number of nodes as:  $3 \cdot 2^{i-3} + 5 \sum_{j=0}^{i-4} 2^j = 2^i - 5 < 2^i$ , again a contradiction.  $\square$

**Lemma 4.** *If there is an RBT with  $2^i$  nodes the root is in position  $N - 2^i k - x + 1$  for  $k = 1, 2, 3, 4$  or  $5$  and  $x = N \bmod 2^i$ .*

*Proof.* There are at most  $5 \cdot 2^i - 5$  nodes in structures of size  $\leq 2^{i-1}$ . All structures of size  $\geq 2^i$  contribute 0 to  $x$ , thus the number of nodes in structures with  $\leq 2^{i-1}$  nodes must be  $x$  counting modulo  $2^i$ . This gives exactly the five possibilities for where the first tree of size  $2^i$  can be.  $\square$

We now describe how to perform an EXTRACTMIN. First, if there is no buffer ( $n \bmod 3 = 0$ ) then DECOMPOSE is executed on the smallest structure. We apply Lemma 4 iteratively for  $i = 0$  to  $\lfloor \log N \rfloor$  and use Lemma 3 to find structures of size  $2^i$ . If there is a structure we call the FINDMIN procedure (i.e. inspect the element of the root node) and remember which structure the minimum element resides in. If the minimum element is in the buffer, it is deleted and the rightmost element is put in the empty position. If there is no buffer, we are guaranteed due to the first step that there is a structure with 1 node, which is now the buffer. On the structure with the minimum element REPLACEMIN is called with the rightmost element of the array. The running time is  $O(\log n)$  for finding all the structures,  $O(\log n)$  for decomposing the smallest structure and  $O(\log n)$  for the REPLACEMIN procedure, in total we get  $O(\log n)$  for EXTRACTMIN.

The INSERT procedure is simpler but the correctness proof is somewhat involved. A new element is inserted in the buffer, if the buffer becomes a node, then the *least significant bit*  $i$  of  $N$  is computed. If at least two structures of size  $2^i$  exist (found using the two lemmas above), then they are linked and become one structure of size  $2^{i+1}$ .

**Lemma 5.** *The INSERT and EXTRACTMIN procedures maintain that at most five structures of size  $2^i$  exist for all  $i \leq \lfloor \log n \rfloor$ .*

*Proof.* Let  $N_{\leq i}$  be the total number of nodes in structures of size  $\leq 2^i$ . Then the following is an invariant for  $i = 0, 1, \dots, \lfloor \log N \rfloor$ .

$$N_{\leq i} + (2^{i+1} - ((N + 2^i) \bmod 2^{i+1})) \leq 6 \cdot 2^i - 1$$

The invariant states that  $N_{\leq i}$  plus the number of inserts until we try to link two trees of size  $2^i$  is at most  $6 \cdot 2^i - 1$ . Suppose that a new node is inserted and  $i$  is not the least significant bit of  $N$  then  $N_{\leq i}$  increases by one and so does  $(N + 2^i) \bmod 2^{i+1}$ , which means the invariant is maintained. Suppose that  $i$  is the least significant bit in  $N$  (i.e. we try to link structures of size  $2^i$ ) and there are at least two structures of size  $2^i$ , then the insertion makes  $N_{\leq i}$  decrease by  $2 \cdot 2^i - 1 = 2^{i+1} - 1$  and  $2^{i+1} - (N + 2^i \bmod 2^{i+1})$  increases by  $2^{i+1} - 1$ , since  $(N + 2^i) \bmod 2^{i+1}$  becomes zero, which means the invariant is maintained. Now

suppose there is at most one structure of size  $2^i$  and  $i$  is the least significant bit of  $N$ . We know by the invariant that  $N_{\leq i-1} + (2^i - (N + 2^{i-1} \bmod 2^i)) \leq 6 \cdot 2^{i-1} - 1$  which implies  $N_{\leq i-1} \leq 6 \cdot 2^{i-1} - 1 - 2^i + 2^{i-1} = 5 \cdot 2^{i-1} - 1$ . Since we assumed there is at most one structure of size  $2^i$  we get that  $N_{\leq i} \leq 2^i + N_{\leq i-1} \leq 2^i + 5 \cdot 2^{i-1} - 1 = 3.5 \cdot 2^i - 1$ . Since  $N \bmod 2^{i+1} = 2^i$  ( $i$  is the least significant bit of  $N$ ) we have  $N_{\leq i} + (2^{i+1} - (N + 2^i \bmod 2^{i+1})) \leq 3.5 \cdot 2^i - 1 + 2^{i+1} = 5.5 \cdot 2^i - 1 < 6 \cdot 2^i - 1$ .

The invariant is also maintained when deleting: for each  $i$  where  $N_i > 0$  before the EXTRACTMIN,  $N_i$  decreases by one. For all  $i$  the second term increases by at most one, and possibly decreases by  $2^{i+1} - 1$ . Thus the invariant is maintained for all  $i$  where  $N_i > 0$  before the procedure. If  $N_i = 0$  before an EXTRACTMIN, we get  $N_j = 2^{j+1} - 1$  for  $j \leq i$ . Since the second term can at most contribute  $2^{j+1}$ , we get  $N_j + (2^{j+1} - ((N + 2^j) \bmod 2^{j+1})) \leq 2^{j+1} - 1 + 2^{j+1} \leq 6 \cdot 2^j - 1$ , thus the invariant is maintained.  $\square$

Correctness and running times of the procedures have now been established.

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